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Experimental Study of Mass Conservation Processes: a Look at the Teaching of Differential Equations from Modeling and Numerical Analysis

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Abstract – *Physical phenomena are typically described in different areas of knowledge using mathematical models. These models permit the assessment of variables associated with the process without building experimental setups. In order to achieve this goal, it is necessary to formulate a differential equation that, depending on the number of variables, can be partial or ordinary type. In this context, the present work evaluates the mass balance for an open system from a numerical and mathematical perspective, and then experimentally validates the results obtained. The analytical solutions of the mathematical models are performed through variable separation, while the numerical solutions are obtained from the fourth-order Runge-Kutta method. The experimental results confirm that the applied differential equation accurately represents the process under consideration and predicts the filling times at which the system reaches the steady-state condition. Furthermore, it describes the system's transient process with an error of less than 4%. Moreover, the influence of operational variables, including the diameters and thermophysical properties of the fluid, on the time required for the system to reach a steady state is quantified numerically. Finally, the present study has proved that how an actual situation reproduced experimentally can be modeled using differential equations and how this strategy facilitates the learning and applying theoretical models in mathematical learning environments.* ***Copyright © 2024 Praise Worthy Prize S.r.l. - All rights reserved.***

Keywords: *Mathematical Model, Ordinary Differential Equation, Mass Balance, Numerical Method, Thermophysical Properties*

Nomenclature

A	Area [m^2]
g	Gravity [m s^{-2}]
h	Step size [s]
i	Iterations number
k	Recurrence factor
L	Liquid level [m]
$\dot{m}$	Mass flow [kg s^{-1}]
n	Data number
t	Time [s]
ρ	Density [kg m^{-3}]
Δ	Difference
Θ	Liquid level at steady state, m
κ	Loss factor
AAD	Absolute Average Deviation
EDO	Ordinary Differential Equation

Subscripts

cal	Computed
e	Input
exp	Experimental
rt	Rectangular tank
s	Output
tc	Circular tank
V	Control volume
vc	Circular tank valve
vr	Rectangular tank valve

I. Introduction

Mathematical models have been employed in many disciplines to elucidate the behavior of variables that govern a phenomenon or physical process. They have had a significant impact on the ongoing transformation of industrial developments and technological advancements [1]. In Europe during the 16th and 17th centuries, the use of mathematical models to establish relationships between the environment and mathematics was a common practice.

This was made possible by the application of mathematical principles to the study of natural phenomena. In this regard, Descartes developed demonstrations to support his position, whereas Galileo, although he held a similar view, placed significant emphasis on experimentation to develop mathematical expressions that would explain his observations [2]. From the nineteenth century to the present day, the use of mathematical models has permitted a remarkable advancement in the representation, control, and modeling of endless processes and systems. The preceding analysis allows for the estimation of not only their costs and

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execution time but also the establishment of methods to optimize them [3], [4]. In the field of mathematical modeling, the relationship between dependent and independent variables in a given process is typically represented by differential equations, which can be either partial or ordinary. These equations can be solved numerically, resulting in either exact or approximate solutions [5], [6]. Numerous works in the literature use mathematical modeling to understand and estimate the performance of multiple systems. For instance, [7] developed a mathematical model to determine the most relevant design parameters for a pilot membrane in an ultrafiltration process. [8] studied the pollutant transport coupled with the biodegradation phenomenon of organic matter using a mathematical model comprising three partial differential equations and their convergence conditions. In addition, the temperature distribution within a passenger carriage was described using differential equations of heat conduction in a non-stationary regime [9]. Recently, [10] formulated a system of transport and concentration equations to evaluate the concentration of Gliadel within the brain. It is noteworthy that not all models incorporate the intricate relationships that govern system or process behavior. Instead, they are flexible analytical tools that offer reasonable computational efficiency [11], [12]. In this context, [13] used two mathematical equations to simulate energy storage in a phase change material and numerically determined the adverse impact of sedimentation on energy conduction.

Similarly, [14] uses numerical and computational techniques to determine mathematical models of chemical kinetics and CO_2 absorption. However, as is the case with numerous models in the literature, these models lack validation with experimental data. While the use of models offers numerous advantages, teaching them presents a significant challenge. This is often attributed to the fact that the results obtained by these models are not validated with real data. Indeed, integrating classroom learning with real-world applications is widely regarded as a crucial aspect of the teaching process, particularly in mathematics. The validation of the results not only enhances students' understanding but also increases the motivation to apply them in personal and professional contexts [15], [16]. In Colombia, the Ministry of National Education (MEN) states, “it is necessary to align learning content with students' daily experiences, contextualizing it within the framework of problematic situations and the exchange of points of view”. Consequently, mathematics takes a broader sense, where students use their mathematical knowledge, modeling, and problem solving in contexts beyond the school environment. This enables students to formulate hypotheses and make decisions to address and adapt to new situations [17]. In the current context, it is essential to align the content of learning with the everyday experiences of learners, and to present and teach it in a context of problem situations and exchange of views [18]. Consequently, the present work describes the mathematical modeling of one of the everyday processes or phenomena in engineering applications, such as mass conservation and, through this problem situation, the aims is to consolidate the learning of these mathematical models through experimentation [19]. For this purpose, mathematical modeling of a mass conservation process for an open system was initially performed [20]. This enabled to predict the time required reaching a permanent regime through a mathematical and numerical approach, i.e., a constant liquid level is reached within the open system.

Subsequently, the discrepancy obtained from these approaches was validated by comparing it with the experimental outcomes. The results demonstrate that for a constant cross-section system, the experimental values are accurately reproduced by the analytical solution of the differential equation, with errors less than 4% in the calculation of the time required for the system to reach a steady state. In the case of a variable cross-section system, a comparison between the experimental and analytical solution was not feasible, as the differential equation model lacks an exact solution. This evidence suggests that numerical solutions are useful in this case. Finally, the results obtained indicate that when the mathematical formulation is used exclusively for predictive purposes, i.e., without taking into account experimental fitting parameters, the theoretical results derived from the mathematical model may diverge from those obtained experimentally by more than 10%.

The article is structured as follows. Section II describes an experimental setup for verifying the height changes of a liquid in an open system. This setup includes a vessel with constant cross-section and a vessel with variable cross-section. Section III presents a mathematical and numerical model for describing mass conservation problems. Section IV discusses the results and their implications. Finally, Section V presents the conclusions of the work and suggestions for future analysis.

II. Methodology

This section provides detailed information regarding the experimental setup employed to validate the changes in liquid level observed in an open system or control volume of fixed cross-section, as well as a control volume of variable cross-section.

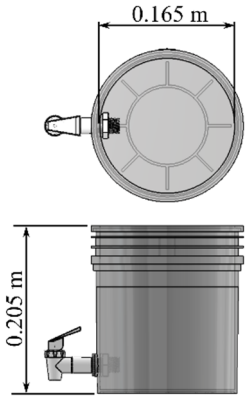
II.1. Test Bed and Experimental Procedure

In general, the experimental procedure conducted in the laboratory involved determining the time at which the height of the liquid level in a container remained constant and unchanging. To this end, a known mass flow was introduced into a tank that was initially empty and provided at the bottom with a valve that allowed the fluid to flow out. The valve dimensions, which influence the mass of water exiting the vessel, result in an increase in the liquid level until it reaches equilibrium, i.e., the system reaches a steady-state. Figs. 1 depict the two vessels utilized during the experiment. The first vessel is represented by a circular tank whose cross-sectional area remains constant.

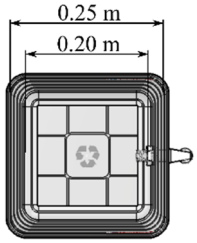
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(a)



(b)





(c)

Figs. 1. (a) Experimental setup and details of the tanks with, (b) constant cross section and (c) variable cross section (units in meters)

In contrast, the second vessel has a square cross-section that varies with the height of the tank. The geometry and dimensions of the tanks to be modeled were defined mathematically and numerically. Water was selected as the working fluid that crosses the boundaries of each system. The indisputable advantage of this selection is the known accuracy of the thermophysical properties to be considered in the mathematical model, which reduces the existence of errors linked to the uncertainty of this variable. Table I shows the most relevant values used during the experimental measurements.

III. Mathematical and Numerical Modeling of the Mass Conservation Problem for a Control Volume

In order to derive the mathematical model that represents the filling process of each container, it is necessary to begin with the mass conservation equation, which establishes that:

$$\left.\frac{dm}{dt}\right|_{VC} = \dot{m}_e - \dot{m}_s \tag{1}$$

The subscripts e and s represent the inflow and outflow of water into and out of each container while the term $\left.\frac{dm}{dt}\right|_{VC}$ portrays the variation of the mass within each control volume. As a result, for the system with a constant cross-section, that is, the circular tank, Eq. (1) can be stated as follows:

$$\left.\frac{dL(t)}{dt}\right|_{VC} = \frac{\dot{m}_e}{\rho A_{tc}} - \frac{\dot{m}_e}{\rho A_{tc}\Theta_{tc}} L(t) \tag{2}$$

The second term on the side of the equality represents the $\dot{m}_s$ expressed in terms of $\dot{m}_s$. The Ordinary Differential Equation (ODE) can be solved analytically by the separation variables method, which yields the following function:

$$L_{tc}(t) = \Theta_{tc}\left[1 - e^{\left[-\frac{\dot{m}_e t}{\rho A_{tc}\Theta_{tc}}\right]}\right] \tag{3}$$

The term Θ in Eq. (3) represents an experimentally obtained measure of the steady-state water level height.

Therefore, it can be reasonably assumed that this function should mathematically predict the time required for the steady state of the process to be reached. A more predictive mathematical model can be derived from the ODEs presented in Eq. (4) and Eq. (5), which avoid the need for prior experimental measurements. Eq. (4) does not take into consideration the energy losses associated with the valve located at the tank outlet. In contrast, in Eq. (5), these losses are considered using the dimensionless factor κ. The objective of both Eqs. (4) and (5) is to incorporate the effects of gravitational forces into the differential equation models.

TABLE I
STATE VARIABLE VALUES FOR WATER USED IN MATHEMATICAL MODELS OF EACH CONTAINER

Variable	Symbol	Value
Water temperature	T_a	23 °C
Atmospheric pressure	P_{atm}	101 kPa
Density	ρ_a	997.6 kg m^{-3}

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It is evident that as the predictive capacity of a mathematical model increases, the complexity of its analytical solution also rises:

$$\left.\frac{dL(t)}{dt}\right|_{VC} = \frac{\dot{m}_e}{\rho A_{tc}} - \sqrt{2g}\ \frac{A_{vc}}{A_{tc}}\ \sqrt{L(t)} \tag{4}$$

$$\left.\frac{dL(t)}{dt}\right|_{VC} = \frac{\dot{m}_e}{\rho A_{tc}} - \kappa \frac{A_{vc}}{A_{tc}} \sqrt{\frac{2g}{1-\left(\frac{A_{vc}}{A_{tc}}\right)^2}}\sqrt{L(t)} \tag{5}$$

In contrast with the linear ODE illustrated in Eq. (2), the Eqs. (4) and (5) are of a nonlinear type. Therefore, an explicit solution for $L(t)$, in terms of time, cannot be obtained. In this case, numerical methods may provide a suitable approximation to the analytical solution of these ODEs (Section III.1.).

In the case of the variable cross-section tank, it was first necessary to obtain an equation relating the height of the liquid level ($L(t)$) to its cross-sectional area. This allowed the subsequent formulation of an energy balance equation for this specific tank type. Fig. 2 provides a detailed illustration of the tank dimensions.

It is crucial to acknowledge that, in the absence of rounding, the cross-sectional area of the tank depicted in Figure 1(c) is a perfect square. In accordance with this consideration, the equation for the mass balance shown in Eq. (6) can be obtained:

$$\left.\frac{d}{dt}\left(A_{rt}L(t)\right)\right|_{VC} = \frac{\dot{m}_e}{\rho} - \frac{\dot{m}_e}{\rho\Theta_{rt}}L(t) \tag{6}$$

A_{rt} term can be derived from Fig. 2 by means of triangle similarity, as follows:

$$A_{rt} = \left(0.2 + 0.167\ L(t)\right)^2 \tag{7}$$

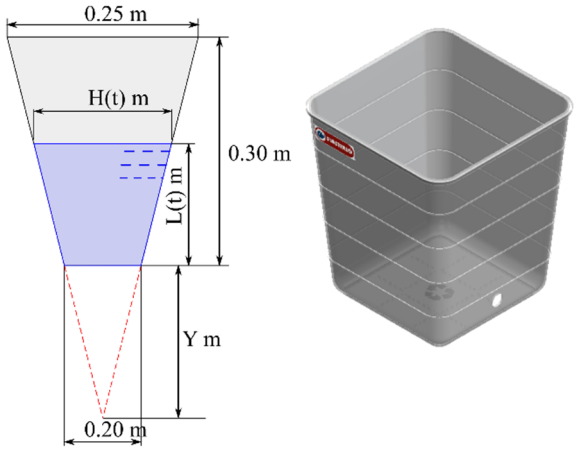


Fig. 2. Side view of the modeled variable cross-section tank. H represents the height-dependent side of the cross section and L is the variable height of the fluid in the tank (units in meters)

This can be substituted into Eq. (7) to obtain:

$$\frac{d}{dt}\left[\left(0.2+0.167\ L(t)\right)^2 L(t)\right]\Big|_{VC} = \frac{\dot{m}_e}{\rho} - \frac{\dot{m}_e}{\rho\Theta_{rt}}L(t) \tag{8}$$

that simplified obtains the following arrangement

$$\left.\frac{dL(t)}{dt}\right|_{VC} = \\ = \frac{1}{\left(0.08366\ L^2(t)\ +\ 0.1336\ L(t) + 0.04\right)} \times \\ \times\left[\frac{\dot{m}_e}{\rho} - \frac{\dot{m}_e}{\rho\Theta_{rt}}L(t)\right] \tag{9}$$

Eq. (9), similar to Eq. (4) and Eq. (5), represents non-linear models of differential equations where the dependent variable (i.e., the time required to reach the equilibrium liquid level in each container) cannot be expressed in a straightforward manner in terms of the independent variable (i.e., time). The term Θ_{rt} in Eq. (9) is fitted with experimental data. However, this ODE can be more predictive by considering the impact of the valve at the outlet of the tank, as was done with the circular cross-section tank in Eq. (4) and Eq. (5). Based on this analogy, Eq. (10) and Eq. (11) were obtained:

$$\left.\frac{dL(t)}{dt}\right|_{VC} = \\ = \frac{1}{\left(0.08366\ L^2(t)\ +\ 0.1336\ L(t) + 0.04\right)} \times \\ \times\left[\frac{\dot{m}_e}{\rho} - A_{vr}\ \sqrt{2g}\ \sqrt{L(t)}\right] \tag{10}$$

$$\left.\frac{dL(t)}{dt}\right|_{VC} = \\ = \frac{1}{\left(0.08366\ L^2(t)\ +\ 0.1336\ L(t) + 0.04\right)} \times \\ \times\left[\frac{\dot{m}_e}{\rho} - \kappa\ A_{vr}\sqrt{\frac{2g}{1-\left(\frac{A_{vr}}{A_{rt}}\right)^2}}\sqrt{L(t)}\right] \tag{11}$$

A_{rt} term in Eq. (11) is given by Eq. (7). It should be noted that both Equations (4), (5) and Equations (9)-(11) are nonlinear and therefore $L(t)$ cannot be expressed analytically as a function of the time variable. An alternative methodology for addressing this challenge is to apply numerical methods to obtain an approximate solution to the problem at hand. The mathematical procedure used to determine the differential equations indicated in Equations (4), (5), (9)-(11) can be found in the

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Appendix section of this manuscript, where it is described in detail.

III.1. Numerical Solution of Nonlinear Ordinary Differential Equations Models

Equations (4), (5) and (9)-(11), being non-linear, do not allow obtaining an explicit expression of L in terms of t.

In this case, numerical methods provide an alternative approach to the analysis of the relationship between liquid level height and filling time. Numerical methods for differential equations are used to find numerical approximations to solutions of Ordinary/Partial Differential Equations (ODE/PDE), also known as "numerical integration" [5]. One of the most widely utilized numerical techniques for numerically solving ODEs with initial conditions is the fourth-order Runge-Kutta (RK) method [21], [22]. In contrast to lower-order RK methods, the RK-4 method exhibits a slight discrepancy with respect to the actual solution of the problem. Additionally, the iterations required to find the numerical solution can be easily programmed. These advantages have been key to its use as a numerical solution technique across numerous disciplines [23]-[28]. The classical RK-4 method exhibits the following format:

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \tag{12}$$

h represents the step size, y_i is the prior value of the dependent variable and the k terms are the recurrence relations which are expressed by:

$$k_1 = f\left(x_i, y_i\right) \tag{13}$$

$$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \tag{14}$$

$$k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right) \tag{15}$$

$$k_4 = f\left(x_i + h, y_i + h\ k_3\right) \tag{16}$$

In this context, the RK-4 method was employed in the present study to solve the nonlinear ODEs derived for each system. For instance, Eq. (4) can be transformed into Eq. (12) as:

$$L_{i+1} = L_i + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \tag{17}$$

and each recurrence function is defined as:

$$k_1 = \frac{\dot{m}_e}{\rho A_{tc}} - \sqrt{2g}\ \frac{A_{vc}}{A_{tc}}\sqrt{L_i} \tag{18}$$

$$k_2 = \frac{\dot{m}_e}{\rho A_{tc}} - \sqrt{2g}\ \frac{A_{vc}}{A_{tc}}\sqrt{\left(L_i + 0.5\ k_1\right)} \tag{19}$$

$$k_3 = \frac{\dot{m}_e}{\rho A_{tc}} - \sqrt{2g}\ \frac{A_{vc}}{A_{tc}}\sqrt{\left(L_i + 0.5\ k_2\right)} \tag{20}$$

$$k_4 = \frac{\dot{m}_e}{\rho A_{tc}} - \sqrt{2g}\ \frac{A_{vc}}{A_{tc}}\sqrt{\left(L_i + \ k_3\right)} \tag{21}$$

A similar methodology was employed for Equations (5), (9)-(11), which were subsequently implemented in Matlab® [29] and the resulting numerical solutions were compared with the experimental data obtained for each tank. Finally, the capacity of the numerical models to anticipate the experimental data was evaluated through the calculation of the Absolute Average Deviation (AAD).

The AAD represents a dispersion indicator and fluctuation between the experimental and calculated values and it is defined as:

$$AAD = \frac{100}{n_e}\sum_{i=1}^{n_e}\left|\frac{\left(L_{calc}\left(t\right) - L_{\exp}\left(t\right)\right)}{L_{\exp}\left(t\right)}\right| \tag{22}$$

n_e represents the number of data used to calculate the error.

IV. Results and Discussions

Depending on the water supply, two mass flow values were used in the experimental measurements. In the case of the circular section tank, the mass flow rate was 0.0551 kg s^{-1}.

This was obtained by averaging five volumetric flow readings and using the water density value presented in Table I. A similar methodology was employed to determine the mass flow of the variable cross-section tank, yielding a value of 0.0480 kg s^{-1} (data for both readings are shown in Table II and Table III).

TABLE II
EXPERIMENTAL FLOW RATE AND MASS FLOW DATA FOR THE TANK WITH CIRCULAR CROSS SECTION

Measurement	Volume [l]	Time [s]	Flow rate [l s^{-1}]	$\dot{m}_{ct}$ [kg s^{-1}]
1	0.8000	14.3900	0.0556	0.0556
2	0.8000	15.1400	0.0528	0.0528
3	0.6000	10.5700	0.0568	0.0568
4	0.7000	12.5600	0.0557	0.0557
5	0.8000	14.6900	0.0545	0.0545
			Average	0.0551

TABLE III
EXPERIMENTAL FLOW RATE AND MASS FLOW DATA FOR THE TANK WITH RECTANGULAR WITH VARIABLE CROSS SECTION

Measurement	Volume [l]	Time [s]	Flow rate [l s^{-1}]	$\dot{m}_{ct}$ [kg s^{-1}]
1	0.8000	17.1400	0.0467	0.0467
2	0.7000	14.1100	0.0496	0.0496
3	0.9000	19.1200	0.0471	0.0471
4	0.8000	16.7500	0.0478	0.0478
5	0.6000	12.3200	0.0487	0.0487
			Average	0.0480

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Knowing the mass flow values for each container, the time for which each system reaches a steady state was determined, i.e., the water height in each tank remains constant. These values are shown in Table IV. In order to estimate the discrepancy in the area computation for the variable cross-section tank with the utilization of Eq. (7), the cross-sectional area as a function of the tank height was determined using the software SolidWorks® [30] and compared with that obtained from Eq. (7). The error observed was less than 1%, which indicates that the Equations (9)-(11) derived from Eq. (7) is an adequate representation. Fig. 3 shows the liquid level variation over time inside the circular section tank obtained analytically using Eq. (3).

It can be observed that the analytical solution closely resembles the transient and permanent moments of filling the vessel. For example, the water level exhibits a significant variation until the100-second mark, after which it shows a relatively less pronounced variation until reaching a value of 7 cm (0.07 m) over a time of 212.6 s.

Subsequently, the level remains constant, thus indicating the attainment of a permanent regime. It is important to highlight that the mathematical model does not incorporate the uncertainties (not considered in this work) associated with the experimental measurements, which leads to the discrepancies obtained between the value estimated by the analytical solution and the experimental data. However, the estimation provided by the mathematical model is in close alignment with the actual outcomes observed in practice, which offers a significant advantage in terms of reducing costs associated with the assembly process and the need for repeated experimentation to predict the results. A limitation of the ODE 3 model is that it necessitates the input of experimental values in order to adjust the parameter Θ.

Consequently, the impact analysis of parameters such as tank diameter, fluid density, mass flow entering the system, area ratio, and resistance coefficient on the time required for the system to reach a steady state is constrained.

TABLE IV
WATER HEIGHT IN STEADY-STATE OBTAINED EXPERIMENTALLY FOR THE CONTAINER WITH CIRCULAR AND VARIABLE CROSS SECTION, RESPECTIVELY

Circular cross-section tank		Rectangular cross-section tank	
Height L [m]	time [s]	Height L [m]	time [s]
0.07 ± 0.001	220.2 ± 0.5	0.15 ± 0.001	1060.2 ± 0.5

TABLE V
LIQUID HEIGHT IN STEADY-STATE OBTAINED EXPERIMENTALLY FOR THE TANK WITH CIRCULAR AND VARIABLE CROSS SECTION, RESPECTIVELY

	SolidWorks			
Liquid height [m]	Cross-sectional area without edge rounding [m^2]	Cross-sectional area with edge rounding [m^2]	Cross-sectional area Eq. (7) [m^2]	Error [%]
0	0.040	0.0397	0.040	0.8649
0.05	0.043	0.0431	0.043	0.8145
0.1	0.047	0.0466	0.047	0.7680
0.15	0.051	0.0503	0.051	0.7269
0.2	0.054	0.0541	0.054	0.6923
0.25	0.058	0.0581	0.058	0.6615
0.3	0.063	0.0622	0.063	0.6323

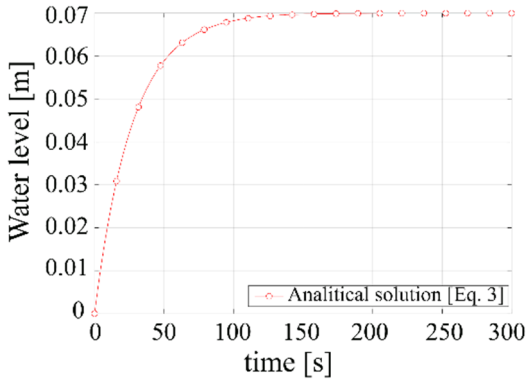
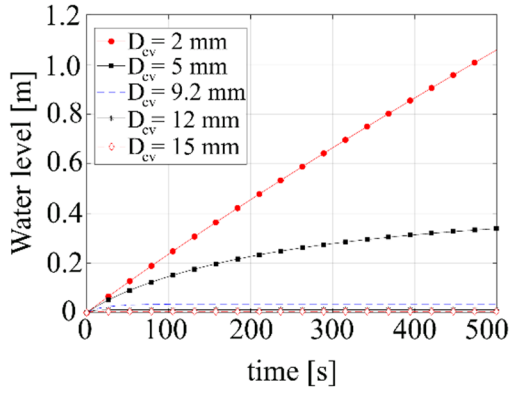
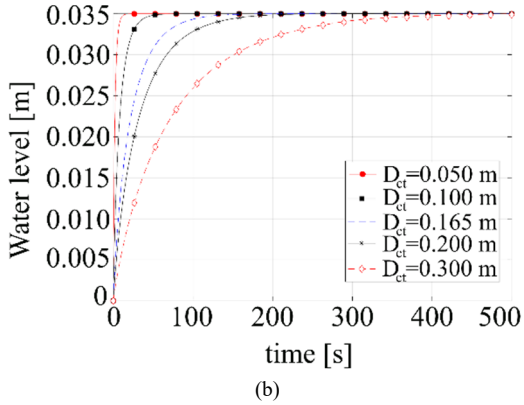


Fig. 3. Analytical solution of Eq. (3) for the circular cross-section tank with Θ=0.07 m

In order to achieve this goal, it may be necessary to utilize more predictive mathematical models, such as Equations (4) and (5), whose solutions are obtained through numerical methods. These models could provide a more comprehensive understanding of the impact of the aforementioned variables. In that sense, Figs. 4 present the impact of the bottom valve diameter and the cross-section diameter for the circular tank on the filling time provided by Eq. (4). For instance, if the diameter of the valve is 2 mm (Fig. 4(a)), the water level in the tank grows linearly, exceeding its height. This implies that a height significantly greater than the current tank height would be required to achieve a steady state (i.e., > 1 m).



(a)



(b)

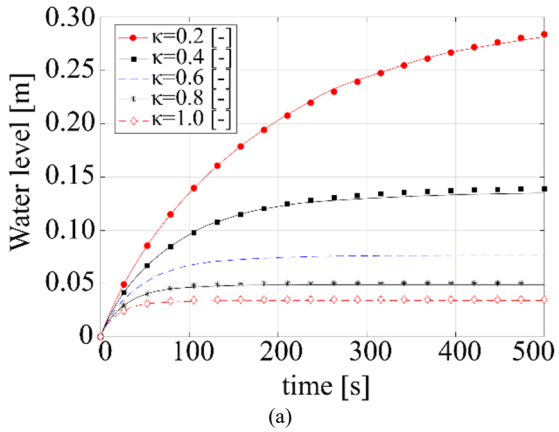
Figs. 4. Impact of bottom valve diameter (a) and tank diameter (b), on the steady-state time for the circular tank with a Δt= 0.1 s

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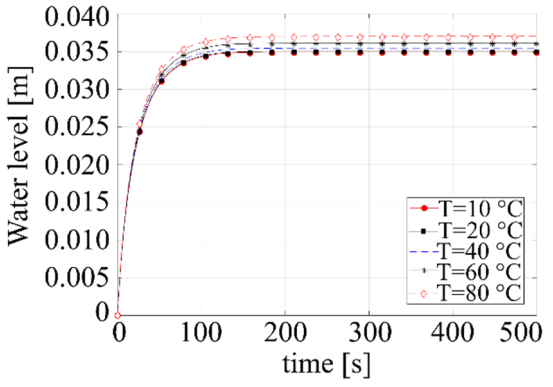
Conversely, the influence of the tank diameter on the steady-state height tends to be relatively less significant.

This is because, for the diameters evaluated, a constant filling height is achieved within a period of less than 500 s. The influence of dimensionless loss factor (κ) and water density on filling time is also analyzed in Eq. (5), with the results presented in Figs. 5. As the valve's loss factor declines, resulting in an increased rate of energy loss from the fluid leaving the vessel, the system takes longer to reach a steady state. This effect is analogous to that observed in the case of small discharge valve diameters (Fig. 4(a)). Conversely, as κ approaches unity, the steady-state time is reached at a value of approximately 212 s, indicating that the current geometrical design of the valve offers minimal resistance in the steady-state condition of the circular tank. Fig. 5(b) illustrates the impact of the fluid density resulting from fluctuations in the water's inflow temperature on the steady-state time of the system.

The impact of this variable on the steady-state condition is relatively insignificant. However, as temperature increases and therefore the density decreases, the liquid level stabilizes at 0.037 m for a time of 229.3 s, i.e., as long as the water temperature is lower than the phase change temperature at ambient pressure (99.61 °C [31]) the water level inside the vessel will not exceed a height of 0.04 m (4 cm). Finally, Fig. 6 presents the numerical results obtained through the application of Equations (9)-(11) to the variable cross-section tank.



(a)



(b)

Figs. 5. Impact of the dimensionless loss factor (a) and fluid density (b) on the steady-state time for the circular tank with a Δt= 0.1 s

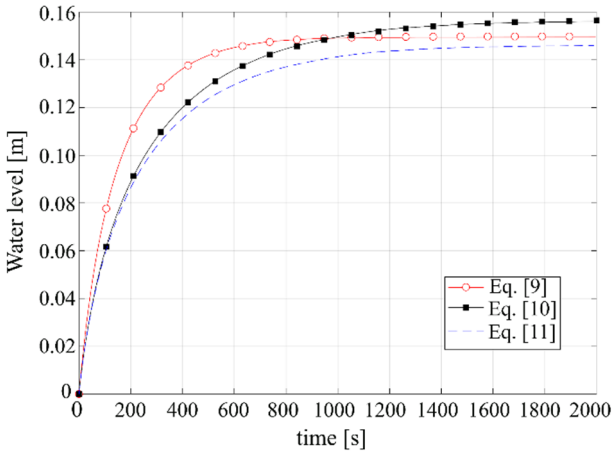


Fig. 6. Numerical solutions with a Δt= 0.1 s for the variable cross-section tank through Equations (9), (10) and (11), respectively

Since the three equation models lack an explicit analytical solution of L in terms of *t*, their solutions were obtained by applying the RK-4 numerical method, with the exception that Eq. (9) has a variable Θ_{rt} that was adjusted experimentally (Θ_{rt}=0.15m). As illustrated in Fig. 6, the calculated time and steady-state liquid level from Eq. (9) yielded values of 1063 s and 0.1494 m, respectively, which are in close approximation to the corresponding values presented in Table V. In contrast, the numerical solution of Equations (10) and (11) indicates that at a time of 1063 s, the liquid level in the vessel reaches a height of 0.1506 m and 0.1421 m, respectively, although the steady-state time is longer. The overall AAD obtained with each model was 6.4% and 8.2%. However, despite the models' full predictive capabilities, which require only the inflow mass flow value to predict the steady-state time, they are highly sensitive to the variation of the tank outlet valve diameter. To illustrate, the precise value of this diameter is 6.0 ± 0.1 mm, and varying this value by 0.1 mm (as indicated in Table V) results in an AAD value approaching 12%, indicating a need for caution when measuring the components included in the mathematical model, especially the discharge diameter of the tank outlet valve.

V. Conclusion

Mathematical models are commonly used in engineering and mathematics to describe physical phenomena. However, validating these models with real data reinforces learning and allows for the exploration of the different variables that contribute to these phenomena.

This can be accomplished without the need for costly laboratory setups. This study is based on the premise that the mathematical modeling of a mass conservation process for a control volume with a constant cross-section and a variable cross-section is required. The principal findings of this study may be summarized as follows. In the context of the constant cross-section system, the results indicated that the mathematical model derived for this system is in good agreement with the experimental data, exhibiting an

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AAD of 3.45%. Additionally, the mathematical model suggests that the outlet diameter of the tank discharge valve and its loss factor are the key variables influencing the time required for steady-state of the system. For the variable cross-section control volume, the ODE that models the mass conservation process does not have an explicit analytical solution for $L(t)$ and therefore, the numerical method of Runge-Kutta was employed to determine the time at which this system reaches a steady state. The numerical solution of this system determines the steady-state condition with an accuracy of 6.4% and 8.2%, respectively. Additionally, because of the non-linear terms utilized for the determination of the system cross-sectional area, the numerical solution becomes particularly susceptible to alterations in the diameter of the system discharge valve. While the ODE models obtained for each control volume estimate the times and heights in a steady state for each system, the coefficients used in these models must be adjusted through experimentation in order to ensure greater accuracy in the mathematical models. The absence of these coefficients may lead to an overestimation of the experimental results, with a potential increase of up to 12% observed in the case of the variable cross-section control volume. Finally, it would be beneficial to consider future analyses that incorporate the inherent uncertainty of the controlling variables within each system. These include variables such as valve diameters and volumetric flows, with the aim of enhancing the precision of the mathematical models derived for each control volume.

Appendix

The mass inside the control volume in Eq. (1) is given by:

$$m = A_{tc} L(t)\ \rho$$

Applying the energy conservation theorem (Bernoulli's Eq. [32]) between the liquid level inside the control volume and the water outlet at the lower valve without consideration of energy losses, the following expression for fluid velocity at the tank outlet can be derived:

$$V_2 = \sqrt{2g}\sqrt{L(t)}A_{vc}$$

Therefore, the mass flow at the outlet ($\dot{m}_s$) employed in Eq. (4) for the tank with a circular cross-section is given by:

$$\dot{m}_s = \underbrace{V_2 A_{vc}}_{\forall}\rho = \sqrt{2g}\sqrt{L(t)}\ A_{vc}\rho$$

Considering the energy losses associated with the discharge valve of each control volume [32], the $\dot{m}_s$ employed in Equations (4) and (5) for the constant cross-section tank is given by:

$$V_1 = V_2\frac{A_{vc}}{A_{tc}};\ \frac{V_2}{2g} = \kappa\sqrt{\frac{2g}{1-\left(\frac{A_{vc}}{A_{tc}}\right)}}\sqrt{L(t)}$$

Similarly, the $\dot{m}_s$ for the variable cross-section tank with and without energy losses associated with the tank discharge valve is given by:

$$\dot{m}_s = A_{vr}\sqrt{2g}\sqrt{L(t)}\rho$$

$$\dot{m}_s = \kappa\ A_{vr}\times \times\sqrt{\frac{2g}{1-\left(\frac{A_{vr}}{\left(0.08366\ L^2(t)\ +\ 0.1336\ L(t)+0.04\right)}\right)^2}}\times \times\sqrt{L(t)}$$

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