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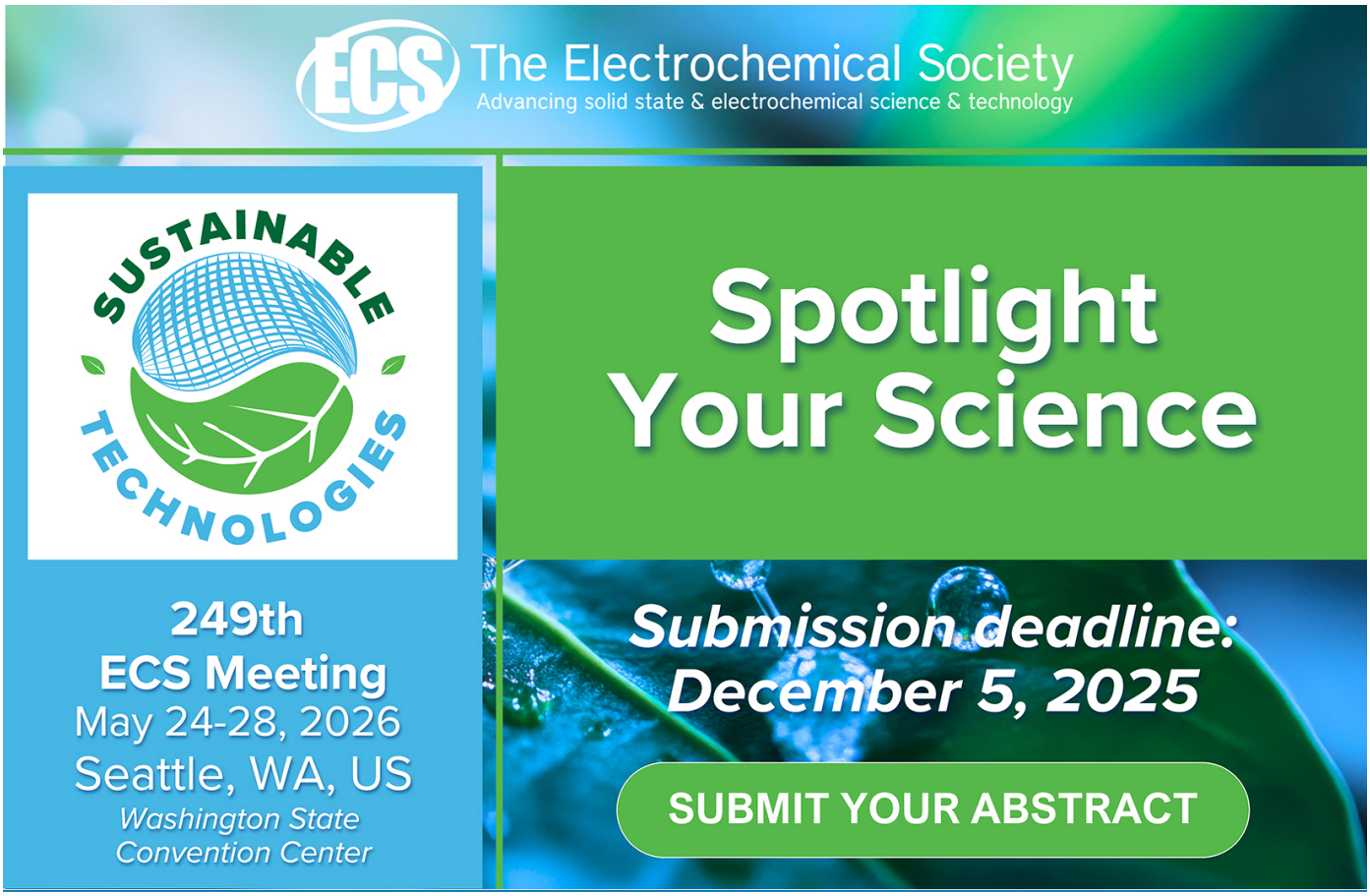
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# Comparison of analytical methods and numerical methods in modeling the physical phenomenon of heat conduction with free radiation

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**Abstract.** The development of mathematical setting for modeling heat conduction phenomena in the presence of radioactive effects is well known. The importance in the study of heat conduction in relation to radioactive effects is relevant in several engineering applications such as combustion, materials science, fluid mechanics and other areas. This research is based on the mathematical model of the heat equation to study the physical phenomenon of heat transfer along a metal bar with slightly insulated sides with the effect of free radiation. To calculate the temperature function that allows modeling the heat transfer process along the bar, the Fourier series solution is constructed step by step, in addition an alternative method of calculating the temperature by using the explicit numerical method is given. The calculation of the temperature along the bar is compared by analytical and numerical method by computing the percentage error and different temperature profiles are plotted to verify the fit of the two approaches. The methods developed throughout the research can be extended to other types of physical phenomena that are useful in related research and in education in engineering subjects such as fluid mechanics and heat transfer.

## 1. Introduction

The study of heat transfer phenomena is related to the effects of conduction, convection, and radiation. The conduction effect refers to the transfer of heat from one region to another by molecular contact, the convection effect is related to the exchange of heat between a medium and a fluid and the radiation effect is related to the emission of heat during a physical process [1]. The applications of the three mechanisms mentioned above range from energetic processes to heat transfer in bricks [2,3], laminar flows in pipes, condensation on vertical tubes [4], radiation between two adjacent surfaces [5].

The analysis of heat transfer with the effect of radiation is regularly modeled by equations of mathematical physics, covering the different modes of radiation, like pure radiation and radiation layer [6]. The use of numerical methods as a tool in the solution of partial differential equations has the advantage of simplicity and fit with traditional methods [7]. In the context of heat transfer with radiation the numerical methods are starting to become relevant [8].

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The importance of this research is based on the modeling of heat transfer in the presence of free radiation, by means of equations adapted from the mathematical physics [9]. The mathematical model has the advantage of being consistent with the classical techniques of Fourier analysis. Throughout the first part of the investigation, the conceptual elements are presented step by step that generate the analytical solution of the differential equation associated with the free radiation phenomenon. Subsequently, the research proposes the application of numerical methods and shows the adjustment of this approach in relation to the analytical solution. The design of the presentation of the different parts of the research should serve as a basis for the teaching of special mathematics in relevant engineering contexts. In addition to the above, the mathematical methods presented in this research can serve as a foundation for future research [10].

## 2. Mathematical modeling

The mathematical model describing heat conduction through a metal bar with free radiation is described by the Equation (1), Equation (2) and Equation (3) [11]. Equation (1) represents the heat equation with the free radiation effect, Equation (2) and Equation (3) describe the boundary conditions and Equation (4) the initial conditions. The constant $\alpha$ is the heat transfer coefficient and $T_a$ is the ambient temperature [11].

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - \alpha(T - T_a), \ 0 < x < \pi, \tag{1}$$

$$T(0,t) = T_a, \tag{2}$$

$$T(\pi,t) = T_a, \tag{3}$$

$$T(0,t) = x + T_a, \tag{4}$$

### *2.1. Analitical solution*

The method for solving the complete Equation (1) to Equation (4) consists of transforming the boundary conditions of the Equation (2) and Equations (3) into a homogeneous system. For this purpose, let $\widehat{T}(x,t) = e^{\alpha t}(T(x,t) - T_a)$, so that $T(x,t) = T_a + e^{-\alpha t}\widehat{T}(x,t)$. Calculating the derivative of the temperature with respect to time and position yields the Equation (5) and Equation (6).

$$\frac{\partial T}{\partial t} = e^{-\alpha t}\frac{\partial \widehat{T}}{\partial t} - \alpha e^{-\alpha t}\widehat{T}, \tag{5}$$

$$\frac{\partial^2 T}{\partial x^2} = e^{-\alpha t}\frac{\partial^2 \widehat{T}}{\partial x^2}. \tag{6}$$

We proceed to the calculation of the boundary conditions and the initial condition for the function $\widehat{T}(x,t)$.The boundary conditions for the auxiliary function $\widehat{T}(x,t)$ are the Equation (7) and Equation (8).

$$\widehat{T}(0,t) = e^{\alpha t}(T(0,t) - T_a) = 0, \tag{7}$$

$$\widehat{T}(\pi,t) = e^{\alpha t}(T(\pi,t) - T_a) = 0. \tag{8}$$

The initial condition for the auxiliary function $\widehat{T}(x,t)$ is the Equation (9).

$$\widehat{T}(x,0) = T(x,0) - T_a = x. \tag{9}$$

Therefore, the function $\widehat{T}(x,t) = e^{\alpha t}(T(x,t) - T_a)$, by Equation (5-6), must verify the equation $\frac{\partial \widehat{T}}{\partial t} = \frac{\partial^2 \widehat{T}}{\partial x^2}$ together with the homogeneous boundary conditions, Equation (7) and Equation (8), and initial condition, Equation (9); by the method of separation of variables [12], the solution of the function $\widehat{T}$ is the Equation (10).

$$\widehat{T}(x,t) = \sum_{n=1}^{\infty} a_n \operatorname{sen}(nx) e^{-n^2 t}, \tag{10}$$

where the coefficients $a_n$ are defined by the Equation (11).

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\operatorname{sen}(nx)dx = \frac{2(-1)^{n+1}}{n}. \tag{11}$$

The analytical solution of the model of heat conduction through a metal bar with free radiation is the Equation (12).

$$T(x,t) = T_a + e^{-\alpha t}\widehat{T}(x,t) = T_a + e^{-\alpha t}\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\operatorname{sen}(nx)e^{-n^2 t}. \tag{12}$$

*2.2. Numerical solution*

The theoretical basis for solving by numerical methods the heat transfer model in the presence of free radiation is to represent the derivative of the temperature by Taylor's theorem [12]. The Equation (13) and the Equation (14) shows this outcome.

$$\frac{\partial T}{\partial t} \sim \frac{T_{i,j+1} - 2T_{i,j}}{k}, \tag{13}$$

$$\frac{\partial^2 T}{\partial x^2} \sim \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(h)^2}. \tag{14}$$

When dividing the spatial interval $(0, \pi)$ into equal parts, the magnitude h represents the width of each subdivision, likewise the k magnitude represents the width in the spatial interval $(0, t)$. The symbol $T_{i,j}$ in Equation (13) and Equation (14) stand for the temperature $T(ih, jk)$.

Substituting the Equation (13) and Equation (14), to calculate the bar temperature function, in complete Equations (1) yields the recursive Equation (15).

$$T_{i,j+1} = \frac{k}{h^2}\left(T_{i+1,j}\right) + \left(1 - \frac{2k}{h^2} - \alpha k\right)\left(T_{i,j}\right) + \frac{k}{h^2}T_{i-1,j} + T_a k. \tag{15}$$

It is possible to calculate the temperature profile by setting the time step jk using the Equation (2), Equation (3), the Equation (4) and the Equation (15).

## 3. Results and discussion

The analysis by numerical methods allows to design a computer program that allows to compare the analytical solution of the Equation (12) with the solution function. It is important to fix $\alpha = 40\ \frac{w}{m}\ ℃$ and $T_a = 2\ ℃$.The behavior results of the relative error in relation to the analytical solution and the solution by numerical methods after 100 and 1000 iterations are displayed in Table 1.

The data in Table 1 show evidence of the convergence of the numerical method implemented from the Equation (15) since the relative error of the temperature profile calculation, after 100 and 1000 iterations, remains small when comparing the analytical solution and the numerical solution. It is possible by means of graphical simulation to verify the fit of the numerical method in relation to the analytical solution. We are going to describe the graphical modeling process. First, we set the time in

the Equation (12) and vary the spatial variable to plotting the temperature. Then the temperature at the spatial positions is calculated for the predetermined time using the Equation (15) and plotted. Finally, the temperature profiles in the analytical and numerical case are compared.

**Table 1.** Relative error of the numerical method.

| Space (cm) | Relative error ($100^{th}$ iteration) | Relative error ($1000^{th}$ Iteration) |
|---|---|---|
| 0.000 | 0.000 | 0.000 |
| 0.314 | 0.014 | 0.080 |
| 0.628 | 0.011 | 0.001 |
| 0.942 | 0.015 | 0.006 |
| 1.256 | 0.020 | 0.007 |
| 1.570 | 0.024 | 0.009 |
| 1.884 | 0.032 | 0.012 |
| 2.199 | 0.039 | 0.015 |
| 2.513 | 0.062 | 0.015 |
| 2.827 | 0.102 | 0.102 |

Figure 1 shows the result of the procedure described above. The blue curve shows the temperature profile after 1000 iterations by the numerical method, the red curve shows the temperature profile at the same instant by the analytical method. The figure 1 shows the behavior of the analytical solution in relation to the numerical solution for the mathematical model of heat conduction with free radiation. As in Table 1, it is possible to observe in figure 1 the good fit of the blue curve in relation to the red curve. The above analysis allows us to conclude that the use of numerical methods is a good tool in the teaching of mathematical to those presented in the research. For the simplicity and accuracy of the simulated results.

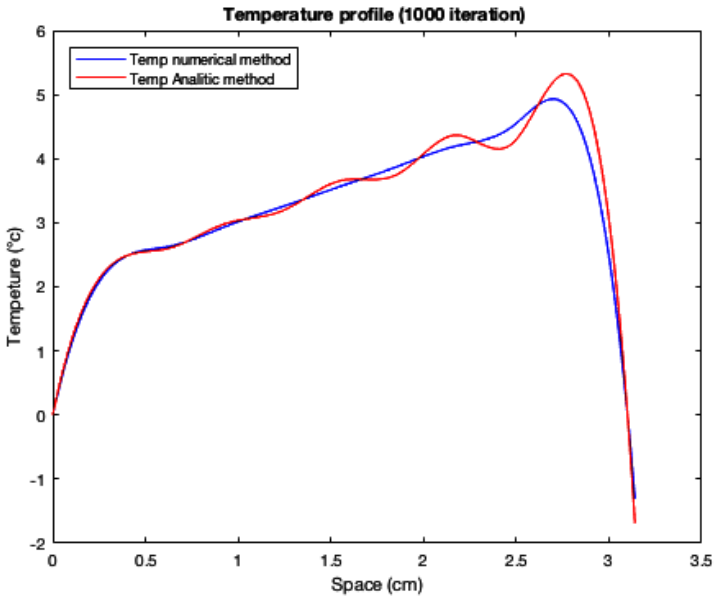



**Figure 1.** Contour of temperature with the numerical method and analitic method.

## 4. Conclusion

The research allowed modeling the phenomenon of heat conduction in the presence of free radiation by means of differential equations. This approach allowed the application of Fourier theory techniques and numerical methods to generate the temperature function. By means of simulation it was possible to compare the analytical and numerical solutions, to conclude the validity of the proposed methods.

## References

[1] Levenspiel O 1984 *Engineering Flow and Heat Exchange* (Boston: Springer)
[2] Nolasco C, Guerrero Gómez G, Gómez J A 2019 Mathematical model of firing process of Ladrillera Ocaña, Colombia *Journal of Physics: Conference Series* **1408(1)** 012017:1
[3] Nolasco C, Jacome N J, Hurtado N A 2019 Solution by numerical methods of the heat equation in engineering applications. Mathematical A case of study: Cooling without the use of electricity *Jornal of Physics: Conference Series* **1388(1)** 012017:1
[4] Kays W M 1980 *Convective Heat and Mass Transfer* (New York: McGraw-Hill)
[5] Stasiek J, Collins M 1993 Radiant and convective heat transfer for flow of a radiation gas in a heated/cooled tube with a grey wall *Journal of Heat and Mass Transfer* **36(14)** 3633
[6] Kim D, Viskanda R 1984 Heat transfer by conduction, natural convection and radiation across a rectangular cellular structure *International Journal of Heat and Fluid Flow* **5(4)** 205
[7] LeVeque R 2007 *Finite Difference Methods for Ordinary and Partial Differential Equations* (Philadelphia: Siam)
[8] Gao D 1999 Numerical solution for a heat conduction problem *Communications in Heat and Mass Transfer* **26(2)** 209
[9] Arpaci V 1966 *Conduction Heat Transfer* (London: Addison-Wesley)
[10] Constanda C 2016 *Solution Technique for Elementary Partial Differential Equations, third edition* (New York: CRC press)
[11] Hillen T, Leonard I H, Van Roessel H 2012 *Partial Differential Equations Conduction Heat Transfer* (New Jersey: John Wiley & Sons)
[12] Carslaw H S, Jaeger J C 1959 *Conduction of Heat in Solids* (New York: Oxford University Press)