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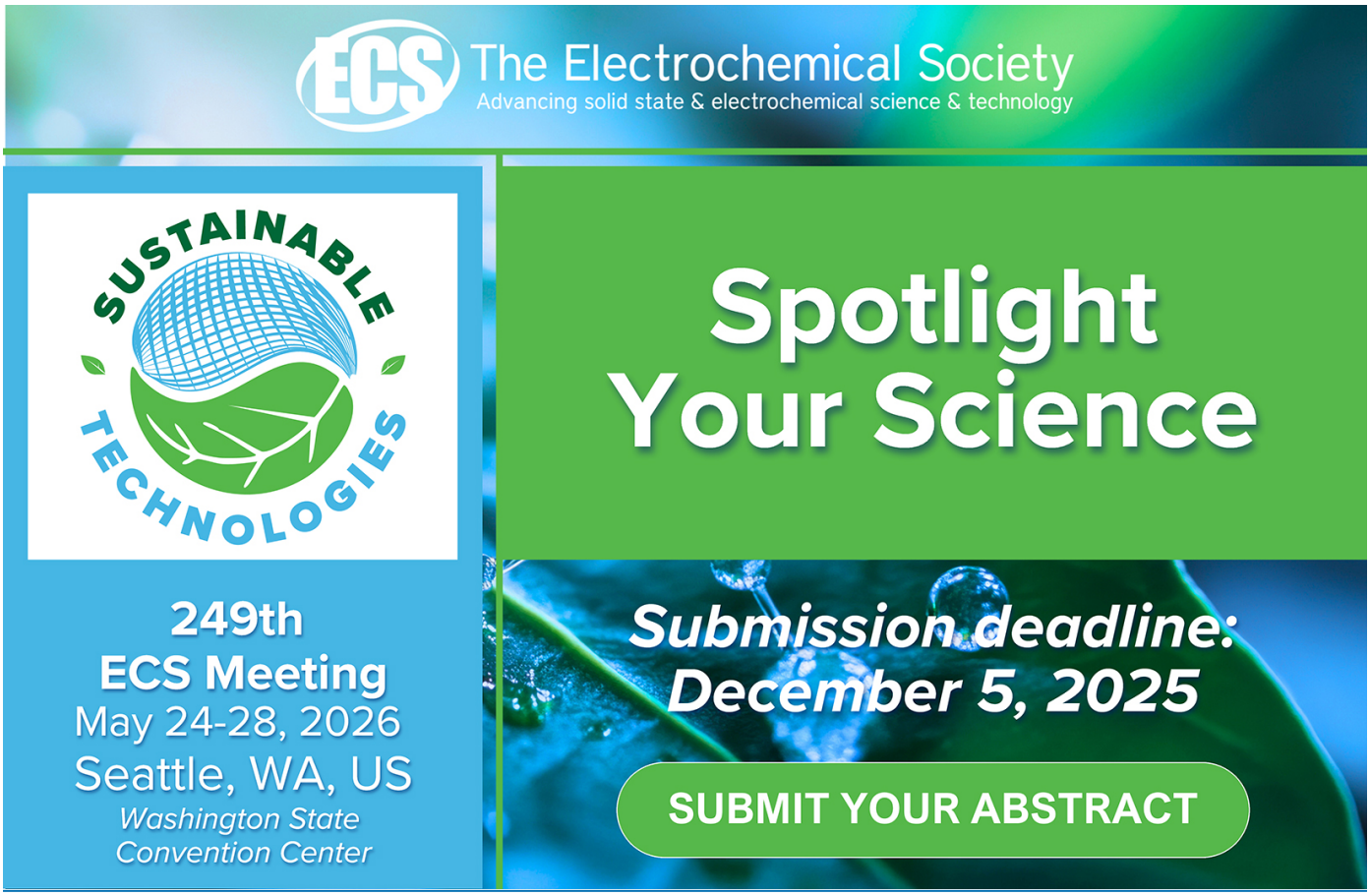
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Modeling of the physical phenomenon of heat generation by using partial differential equations

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Abstract. The equations of mathematical physics are a natural environment for modeling physical phenomena, an example of the above is evidenced by the heat equation in relation to its use in a variety of applications; directly related to the equations of mathematical physics are the solution methods that are used to construct the predictive models. This paper describes step by step the analytical method of separation of variables to perform a complete description of the heat conduction phenomenon in the presence of a heat generation source. The investigation by using mathematical arguments allowed to calculate the temperature function as the addition of a Fourier series and a function which represents the steady state; by performing a computational simulation, it was possible to demonstrate the accuracy of the results achieved.

1. Introduction

The relationship between differential equations and trigonometric series has its origin in Euler's work related to planetary dynamics. Fourier's theory on trigonometric series was originally formulated to study problems related to string vibrations and heat conduction [1]. Fourier theory has been developed from several areas of research such as the analysis of the convergence of trigonometric series [2], the study of special functions [3], and applications to the equations of mathematical physics [4].

The Fourier series can be applied to welding processes because the thermal cycles produced by the heat source that causes change in the physical state and metallurgical transformations. This can generate imperfect microstructures, residual stresses, plastic deformations, and discontinuities in both the base material and the filler material during the rapid solidification phase [5]. Mathematical modeling of heat flow in welding processes using Fourier series can become an excellent mathematical tool for analyzing thermal data and systematic investigations of thermal characteristics of any welding process [6].

This research has as initial contribution to apply the Fourier theory to solve a mathematical model that represents the heat transfer in the presence of a heat source.This approach has the relevance of formulating a relevant physical phenomenon in the research as well as providing a promising environment to show the use of Fourier's theory [7-9].In the research the application of the Fourier theory is studied in depth by presenting all the steps to calculate the solution of the mathematical model by trigonometric series.In addition to the above, the phenomenon of heat conduction with heat source is characterized by calculating the steady-state temperature function by means of mathematical arguments and computational simulation.

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2. Mathematical modeling using Fourier theory

The mathematical model that characterizes the physical phenomenon of conduction with heat source [10] is represented in the Equation (1), Equation (2) and Equation (3).

$$\frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial x^2} + g(x), 0 < x < L,\ t > 0, \tag{1}$$

$$T(0,t) = T_1, T(L,t) = T_2, \tag{2}$$

$$T(x,0) = f(x),\ 0 < x < L. \tag{3}$$

The function T represents the temperature, the variable x the position and the function represent $g(x)$ the heat source. The Equation (2) represents the boundary conditions, and the Equation (3) represents the initial condition.

2.1. Homogeneous model

In order to solve the mathematical model represented by the Equation (1) to Equation (3), is studied the homogeneous system presented in the Equation (4), Equation (5) and Equation (6) modeling heat transfer without additional effects with isolated ends [11].

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, 0 < x < L,\ t > 0, \tag{4}$$

$$T(x,0) = 0, T(L,t) = 0, t > 0, \tag{5}$$

$$T(x,0) = f(x),\ 0 < x < L. \tag{6}$$

Applying the separation of variables [12] to the temperature function we arrive at $T(x,t) = X(x)u(t)$ and the Equation (7).

$$\frac{X''}{X} = \frac{u'}{c^2 u} = -\lambda, \tag{7}$$

where λ is a constant called the eigenvalue of the partial differential Equation (4). Equation (7) together with the boundary conditions leads to the Equation (8) and the Equation (9).

$$X''(x) + \lambda X = 0,\ X(0) = 0,\ X(L) = 0, \tag{8}$$

$$u'(t) + c^2 \lambda u(t) = 0. \tag{9}$$

The boundary value Equation (8) is an eigenvalue problem [9], its nontrivial solutions, except for a constant of proportionality, are the eigenfunctions in Equation (10).

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right),\ n = 1,2,3, ..., \tag{10}$$

associated with the eigenvalues in Equation (11).

$$\lambda_n = \frac{n^2\pi^2}{L^2},\ n = 1,2,3, ... \tag{11}$$

Substituting the complete Equation (11) into the first-order ordinary differential Equation (9) yields except for a constant of proportionality the Equation (12).

$$u_n(t) = \exp\left(-\frac{n^2\pi^2c^2t}{L^2}\right),\ n = 1,2,3,\ldots \tag{12}$$

Accordingly, to the Equation (10) and Equation (11), the temperature function that verifies the heat conduction phenomenon with homogeneous boundary conditions is the Equation (13).

$$T_n(x,t) = \exp\left(-\frac{n^2\pi^2c^2t}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right), \qquad n = 1,2,3,\ldots \tag{13}$$

The superposition principle [13] allows us to express the temperature function as follows in Equation (14).

$$T(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2c^2t}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right). \tag{14}$$

The Equation (14) satisfies the complete Equation (4) and Equation (5). The temperature function $T(x,t)$ must verify the Equation (6), therefore by replacing $t = 0$ in complete Equation (14) is held the Equation (15).

$$T(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x). \tag{15}$$

The coefficients c_n in the Equation (15) are determined by the Fourier's theory [2] by the integral in Equation (16).

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{16}$$

2.2. Non homegeneous model

To solve the non-homegeneous problem of the Equation (1) to Eqaution (3) it is possible to assume the Equation (15).

$$T(x,t) = v(x) + w(x,t), \tag{17}$$

being $v(x)$ the steady state solution and $w(x,t)$ the transient solution [9]. The dependence of the stable function allows us to assume that the function $v(x)$ must satisfy the Equation (18).

$$c^2v''(x) + q(x) = 0,\ \ v(0) = T_1,\ \ v(L) = T_2. \tag{18}$$

The transient solution $w(x,t)$ verifies the homogeneous problem of heat conduction in Equation (19), Equation (20), and Equation (21).

$$\frac{\partial w}{\partial t} = c^2\frac{\partial^2 w}{\partial x^2}, 0 < x < L,\ \ t > 0, \tag{19}$$

$$w(x,0) = 0, w(L,t) = 0,\ \ t > 0, \tag{20}$$

$$w(x,0) = f(x) - v(x),\ \ 0 < x < L. \tag{21}$$

By the Equation (14), Equation (15), and Equation (16), the function $w(x,t)$ is defined by the Equation (22).

$$w(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{n^2\pi^2c^2t}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right), \tag{22}$$

where the coefficients $\mathbf{c_n}$ are determined in the Equation (23).

$$c_n = \frac{2}{L}\int_0^L (f(x) - v(x)) \sin\left(\frac{n\pi x}{L}\right) dx. \quad (23)$$

3. Results and discussion

To verify the procedure outlined in the preceding section it is possible to replace the parameters and functions by $c = 1,\ q(x) = 0.6x,\ L = 10,\ T_1 = 5,\ T_2 = 15,\ f(x) = 11x + 5$. Substituting the above parameters into the Equations (1-3) allows to generate the mathematical model for the conduction phenomenon with heat generation represented in the Equation (24), Equation (25), and Equation (26).

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + 0.6x, 0 < x < 10,\ t > 0, \quad (24)$$

$$T(x, 0) = 5, T(L, t) = 15, t > 0, \quad (25)$$

$$T(x, 0) = 11x + 5,\ 0 < x < L. \quad (26)$$

By solving the second order Equation (18) that gives rise to the steady state function $v(x)$ in Equation (27).

$$v(x) = -0.1x^3 + 11x + 5. \quad (27)$$

By applying the Equation (23), it is possible to generate the coefficients c_n in the Equation (28).

$$c_n = 0.2\int_0^{10} 0.1x^3 \sin\left(\frac{n\pi x}{L}\right) dx = \frac{200}{n^3\pi^3}(6(-1)^n - n^2\pi^2(-1)^n). \quad (28)$$

Finally, to calculate the temperature function $T(x, t)$, the Fourier coefficient c_n Equation (27) and the function $v(x)$ Equation (28) are replaced in complete Equation (17). The above procedure allows to build up the expected result in the Equation (29).

$$T(x, t) = -0.1x^3 + 11x + 5 + \frac{200}{\pi^3}\sum_{n=1}^{\infty} \frac{(6(-1)^n - n^2\pi^2(-1)^n)}{n^3} \exp\left(-\frac{n^2\pi^2 t}{100}\right) \sin\left(\frac{n\pi x}{10}\right). \quad (29)$$

From the Equation (27) it is possible to verify the Equation (30), since $\lim_{n\to\infty} w(x, t) = 0$.

$$\lim_{n\to\infty} T(x, t) = v(x) = -0.1x^3 + 11x + 5. \quad (30)$$

By means of simulation, it is possible to verify Equation (29) to establish the temperature function is T as a function dependent on the variable x. Figure 1 shows the temperature profile for different instants (the blue curves) in conjunction with the steady-state (the red curve); the blue curves represent the transient state temperature behavior, which is represented in Equation (22), the red curve represents the function v(x). Figure 1 demonstrates the convergence of the series of Equation (27) as well as the behavior of the physicist for which the steady state is the limit of the transient state [14].

The methodology for calculating the temperature function that allows modeling the physical phenomenon of conduction was defined step by step with the intention of serving as a basis for teaching advanced courses in engineering careers involving thermal conduction phenomena. It also allows students to recall concepts from calculus and differential equations in a useful engineering context.

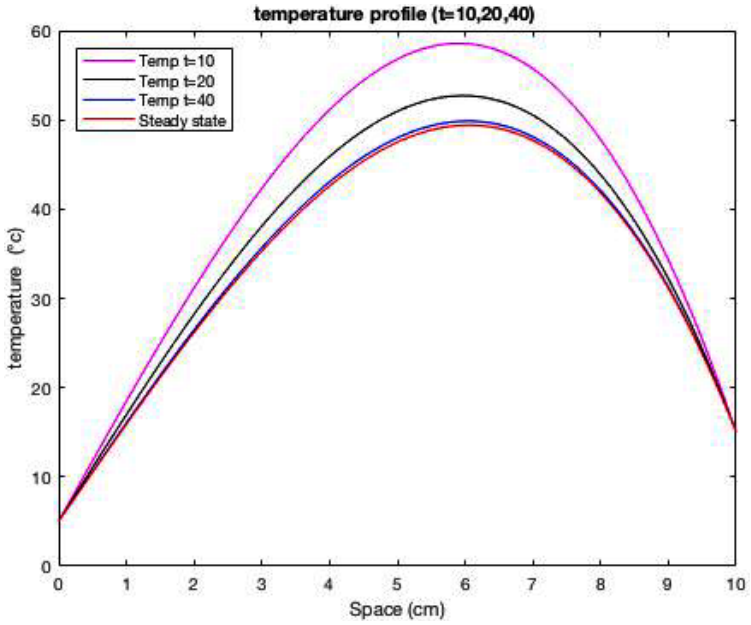


Figure 1. Temperature profile with Fourier series.

4. Conclusion

This research presents an application of conduction in the presence of a heat source. By means of a detailed development, the function that describes the temperature in general is calculated and then the calculation is used in a particular case. The mathematical analysis and simulation allowed to demonstrate that the steady state is the limiting case of the transient state. This coincides with the classical theory of heat conduction. The detailed development of the calculation of the temperature function allowed the definition of a step-by-step method with the advantage of being able to be used in teaching the use of Fourier theory in the context of modeling situations involving heat conduction.

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