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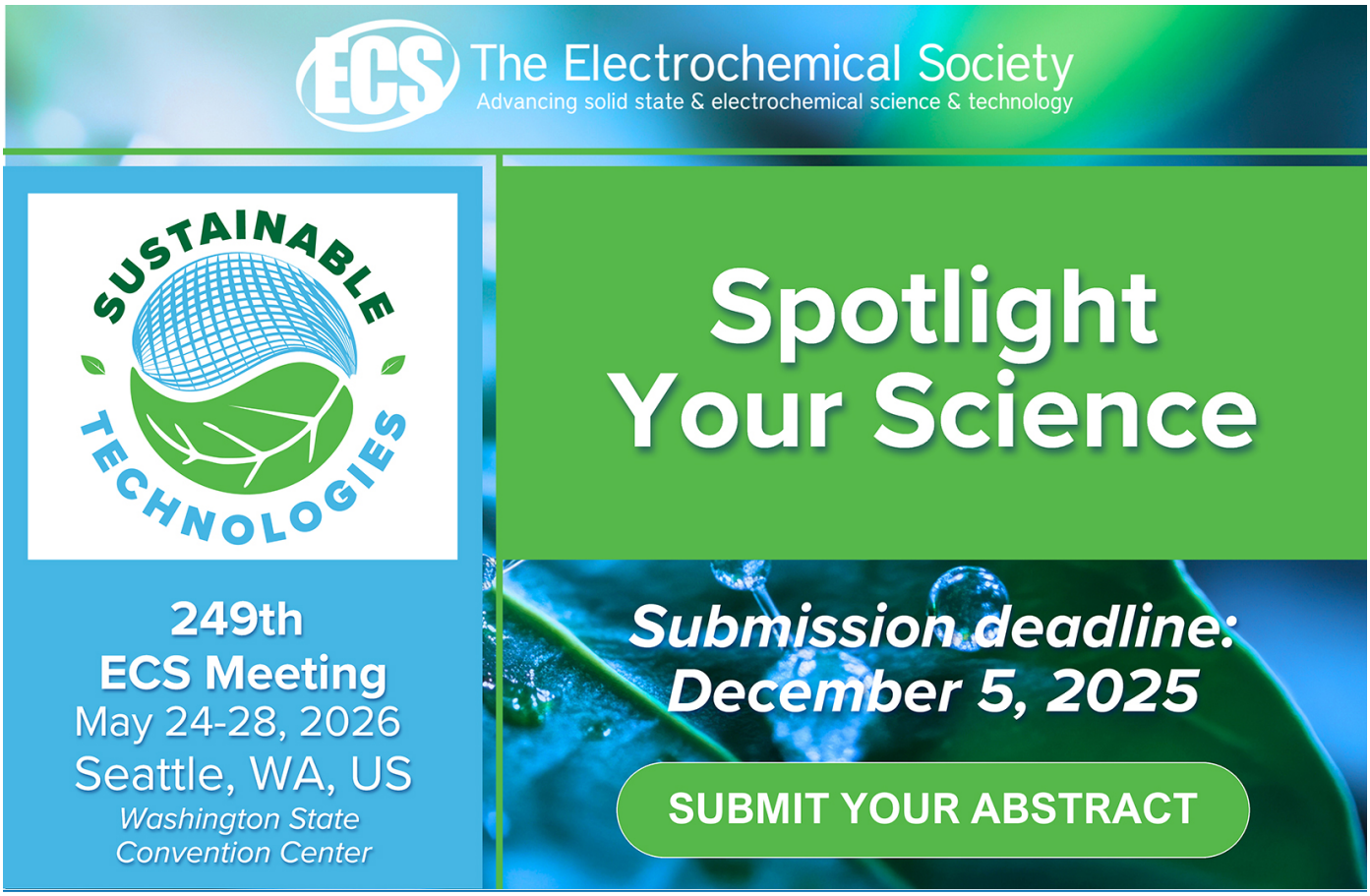
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Applications of differential equations to model the physical phenomenon of heat transfer with an internal energy source

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Abstract. The mathematical modeling of the physical phenomenon of conduction in the presence of heat sources has various uses in engineering ranging from heat diffusion, pollution generation in large cities and chemical reactions. From the mathematical theory, the study of physical phenomena with the presence of heat is approached from techniques of polynomial approximation, analysis of variations and numerical methods. In this research a mathematical model is proposed to study the heat conduction in a metal bar with a heat source. In the first part of the work, by the application of the method of the parameter variation, the exact solution of the mathematical model is calculated. Subsequently, by means of a case study of the physical phenomenon of heat conduction with internal energy source, with the use of computational simulation, the convergence of the Fourier series linked to the physical phenomenon is shown. The mathematical method proposed in the paper can be applied to other physical phenomena such as wave propagation, electric potential, and chemical reactions.

1. Introduction

The study of mathematical models to describe the physical phenomenon of heat conduction with a heat source is relevant to applications and mathematical theory. Applications of the heat equation in the presence of heat sources are related to heat transfer, transport problems and hydrology [1]. Another application has to do with pollution sources in densely populated cities [2]. In the context of biochemistry applications, the additional term in the heat equation can be interpreted as the source of a chemical reaction [3].

From the point of view of mathematical theory, the techniques for studying the model vary from the solution by polynomial functions [4], methods of variational analysis [5] and numerical methods [6]. The mathematical method proposed in the work is an extension of the method of parameter variation for equations in several variables, which is compatible with Fourier theory and allowed to generate an analytical solution for the physical phenomenon of heat transfer [7].

The research proposes a model derived from the heat equation to study the termal conduction process in the presence of a position- and time-dependent heat source [8]; initially, a relevant contribution of the work consists in applying a simple technique, the parameter variation, for solving ordinary differential equations to modeling with the heat equation [9]. Subsequently, by careful use of Fourier analysis [10],

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the temperature function is calculated. In order to validate the procedure used in the first part of the research, the fourier series associated with heat conduction phenomena with energy source is computed and the convergence of the Fourier series predicting the temperature is verified by simulation [11]. The method proposed in the research can be used to model physical phenomena such as fluid dynamics [12].

2. Mathematical modeling

Let us consider a thin bar with zero temperature at the ends. The mathematical model that characterizes the physical phenomenon of conduction with heat generation in variables of space (x) and time (t) is represented by Equation (1), Equation (2) and Equation (3) [8].

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2} + h(x,t), 0 < x < L, \quad t > 0. \tag{1}$$

The function T represents the temperature, and the function $h(x,t)$ represents the heat generation. Equation (2) represents the boundary conditions and Equation (3) represents the initial condition.

$$T(0,t) = 0, \quad T(L,t) = 0, \quad t > 0, \tag{2}$$

$$T(x,0) = f(x), \quad 0 < x < L. \tag{3}$$

The following method is very similar to the parameter variation method [7] used to solve a non-homogeneous linear ordinary differential Equation (4).

$$y''(x) + p(x)y'(x) + Q(x)y(x) = q(x). \tag{4}$$

Consider the associated homogeneous (Equation (5)), and the two linear solutions $y_1(x)$ and $y_2(x)$, the principle of superposition guarantees that the linear combination $c_1y_1(x) + c_2y_2(x)$ is also a solution of the Equation (5), with parameters c_1 and c_2 [7]. The solution of the Equation (4) is obtained by varying the parameters, that is, by assuming an expression of the form $c_1(x)y_1(x) + c_2(x)y_2(x)$. To determine the functions $c_1(x)$ and $c_2(x)$ a procedure must be carried out in which the function $q(x)$that appears in the Equation (4) is very important. The mathematical model of the research (Ecuation (1)) is associated with the homogeneous Equation (6).

$$y''(x) + p(x)y'(x) + Q(x)y(x) = 0, \tag{5}$$

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2}, 0 < x < L, \quad t > 0, \tag{6}$$

By applying the principle of superposition and the method of parameter variation, an expression of the form Equation (7) is computed.

$$\sum_{n=1}^{\infty} c_n(t) \ \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1,2,3,\ldots \tag{7}$$

The eigenvalues and eigenfunctions [10] when the function $h(x,t) = 0$ are calculated by the method of separation of variables and are respectively stated in Equation (8).

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \ \Phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1,2,3,\ldots \tag{8}$$

The method of variable separation allows to assume that the solution of the mathematical model has the form of the Equation (9).

$$T(x,t) = \sum_{n=1}^{\infty} c_n(t)\ \Phi_n(x). \tag{9}$$

Deriving the Equation (9) term by term leads us to the Equation (10).

$$\Phi_n''(x) + \lambda_n \Phi_n(x) = 0. \tag{10}$$

Substituting the complete Equation (10) in Equation (1) produces the mathematical expression Equation (11).

$$\sum_{n=1}^{\infty} c_n'(t)\Phi_n(x) = -\alpha^2 \sum_{n=1}^{\infty} c_n(t)\lambda_n \Phi_n(x) + h(x,t). \tag{11}$$

The above series is equivalent to the Equation (12).

$$\sum_{i=1}^{n}\left(c_n'(t) + \alpha^2 \lambda_n c_n(t)\right)\Phi_n(x) = h(x,t). \tag{12}$$

By multiplying the Equation (12) by $\Phi_n(x)$, subsequently calculating the integral in over the interval $[0, L]$ and bearing in mind the orthogonality of the functions $\Phi_n(x)$ in $[0, L]$, the Equation (13) is deduced.

$$\left(c_m'(t) + \alpha^2 \lambda_m c_m(t)\right) \int_0^L \Phi_m^2\,(x)dx = \int_0^l h(x,t)\Phi_m(x)dx. \tag{13}$$

As $\int_0^L \Phi_m^2\,(x)dx = \frac{L}{2}$ and when substituting m for n in Equation (13) we obtain the Equation (14).

$$\left(c_n'(t) + \alpha^2 \lambda_n c_n(t)\right) = \frac{2}{L}\int_0^l h(x,t)\Phi_n(x)dx\,, t > 0, n = 1,2,3,\ldots \tag{14}$$

The boundary conditions (Equation (2)) are verified as $\Phi_n(x)$ it does. Substituting $t = 0$ into Equation (9), the initial condition of Equation (3) is obtained in Equation (15).

$$T(x,0) = \sum_{n=1}^{\infty} c_n(0)\ \Phi_n(x) = f(x). \tag{15}$$

If we again multiply Equation (15) and integrate over the interval $[0, L]$, considering the orthogonality, we arrive at the Equation (16).

$$\int_0^L f(x)\ \Phi_m(x)dx = c_m(0) \int_0^L \Phi_m^2\,(x)dx. \tag{16}$$

From Equation (16) it is possible to conclude the Equation (17).

$$c_n(0) = \frac{2}{L}\int_0^L f(x)\ \Phi_n(x)dx. \tag{17}$$

To finalize the calculation of the temperature function $T(x, t)$ that verifies the Equation (9), it is stated that the function $c_n(t)$ must verify the initial value problem (Equation (18) and Equation (19)).

$$c_n'(t) + \frac{\alpha^2 n^2 \pi^2}{L^2} c_n(t) = \frac{2}{L}\int_0^L h(x,t)\sin\left(\frac{n\pi x}{L}\right)dx, \tag{18}$$

$$c_n(0) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{19}$$

3. Results and discussion

The method described in the previous section allows by its general nature to be applied to a diverse variety of physical phenomena [13]. A particular case with concrete physical parameters is shown below. To verify the procedure outlined in the preceding section it is possible to assume $\alpha^2 = 1$, $h(x,t) = xte^{-t}$, $L = \pi$, $f(x) = x$. Substituting the above parameters into the Equation (1) to Equation (3) allows to generate the mathematical model for the conduction phenomenon with heat generation represented in the Equation (20), Equation (21) and Equation (22).

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + xte^{-t}, 0 < x < \pi, \quad t > 0, \tag{20}$$

$$T(0,t) = 0, \quad T(L,t) = 0, \quad t > 0, \tag{21}$$

$$T(x,0) = x, \quad 0 < x < \pi. \tag{22}$$

The Equation (9) allows us to express the temperature function by means of Equation (23).

$$T(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1,2,3,\ldots \tag{23}$$

The Equation (18) implies that the functions $c_n(t)$ satisfy the initial value problem (Equation (24) and Equation (25)).

$$c_n'(t) + n^2 c_n(t) = \frac{2}{\pi}\int_0^{\pi} xte^{-t}\sin(nx)dx, \tag{24}$$

$$c_n(0) = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx. \tag{25}$$

Solving the integrals of the Equation (24) and Equation (25) generates the expressions Equation (26) and Equation (27).

$$c_n'(t) + n^2 c_n(t) = \frac{2(-1)^{n+1}te^{-t}}{n}, \tag{26}$$

$$c_n(0) = \frac{2(-1)^{n+1}}{n}. \tag{27}$$

Finally, the function is calculated $c_n(t)$ by integration in Equation (27) and the explicit temperature solution is generated Equation (28).

$$T(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\left((n^2-1)te^{-t} - e^{-t} + (n^4 - 2n^2 + 2)e^{-n^2 t}\right)\sin(nx). \tag{28}$$

By simulation [11] it is possible to demonstrate the convergence of Equation (28). Figure 1 shows the temperature profile with a fixed value for $t = 0.1$ and $n = 2$ (blue curve), $n = 3$ (red curve), $n = 999$ (black curve). The Fourier theory [10] guarantees the convergence of the series, Equation (28), through mathematical analysis, from Figure 1 it is possible to guarantee the convergence of the series when $n = 3$ (red curve) is very close to the $n = 999$ (black curve). In addition, by means of the solution exact when $n = 999$ it is we have the steady state Equation (29), $T(x)$, of the conduction phenomenon with heat generation.

$$T(x) = \sum_{n=1}^{999} \frac{2(-1)^{n+1}}{n}\left((n^2-1)te^{-t} - e^{-t} + (n^4 - 2n^2 + 2)e^{-n^2 t}\right)\sin(nx). \tag{29}$$

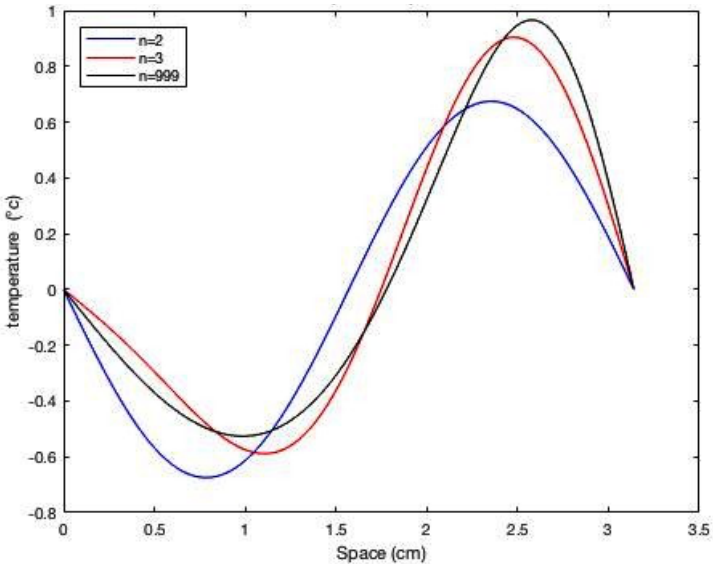


Figure 1. Contour of temperature with the analitic method.

4. Conclusion

This research allowed the study of the physical phenomenon of heat transfer with a heat source by modeling with differential equations. This approach allowed by means of the technique of parameter variation in conjunction with the Fourier theory to calculate a general solution of the mathematical model by means of a series to calculate the temperature function. To validate the theoretical results the problem of heat conduction in a bar with known parameters is formulated, by means of simulation verifies the convergence of the series and the adequacy of the result with the generation of the curve that represents the stable state of the physical phenomenon. The solution technique of the work can be extended to wave and magnetic phenomena.

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