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Modeling of heat transport phenomena using the equations of mathematical physics

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Abstract. The study of physical phenomena that include conservative principles is part of the research field of the equations of mathematical physics. To deepen in methods to solve the equations of mathematical physics is a contribution in understanding the modeling of applications in different areas. This research studies the physical phenomenon of heat transport with convection from the viewpoint of modeling with differential equations. The advantage of working with equations is to apply the techniques of mathematical analysis and numerical methods to obtain the temperature function. In the research, the solution of the heat transport model is computed according to the analytical method of separable variables in order to represent the temperature function as a trigonometric series. With the help of a simple numerical method, it is possible to derive a scheme of calculation of the temperature function. By performing a case study, the methods are compared, and their fit is verified by simulation.

1. Introduction

The scope of physical phenomena that are suitable for modeling by using mathematical physics equations is extremely wide; an example of the above is the study of dissipative waves, problems in continuum mechanics, the Fisher equation in biology, fluid dynamics, among others [1]. A prominent place in the study of the equations of mathematical physics is reserved to the diffusion equation [2]; a particular case of physical phenomena involving the diffusion model are chemical concentrations, temperature change, material properties and energy transport [3].

In this research, the phenomenon of heat transport with convection is studied with modeling from differential equations; through a detailed development supported by analytical methods [4,5] the temperature function is computed; the outlined steps can be useful in further investigations together with the teaching in science and engineering departments. In contrast to the analytical method, the transport equation is solved with the explicit numerical method [6,7] which is very easy and straightforward. In the spirit of validating the analytical and numerical calculation of the temperature function with the use of simulation, the fit of the methods is verified.

From the numerical method proposed in the research it is possible to carry out a study of the modeling of transport phenomena by using the finite elements [8], with the finality of deepening in relevant theoretical aspects relevant to the study of partial differential equations [9].

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2. Mathematical model

The differential equation describing the physical behavior of heat transport with convection is defined by the Equation (1) [10].

$$\frac{\partial H}{\partial t} = \alpha \frac{\partial^2 H}{\partial x^2} - C \frac{\partial H}{\partial x}, 0 < x < a, \tag{1}$$

where the function H represents the temperature, α the specific heat and C the convective velocity. The Equation (2) and Equation (3) correspond to the boundary conditions and Equation (4) with the initial condition.

$$H(0,t) = 0, \tag{2}$$

$$H(a,t) = 0, \tag{3}$$

$$H(x,0) = f(x). \tag{4}$$

2.1. Analytical solution

To calculate the temperature function $H(x,t)$, let us assume that it is possible to separate the variables as indicated in the following Equation (5) [11].

$$H(x,t) = v(x)w(t). \tag{5}$$

From the Equation (1) and Equation (5) it is possible to set the Equation (6), Equation (7), and Equation (8).

$$v''(x) - \frac{C}{\alpha}v' + \lambda v(x) = 0,\ w'(t) + \lambda\alpha w(t) = 0, \tag{6}$$

$$v(0) = 0, \tag{7}$$

$$v(a) = 0. \tag{8}$$

Through the change of variable as described in Equation (9).

$$z(x) = e^{\frac{-Cx}{2\lambda}} v(x), \tag{9}$$

the function $z(x)$ satisfies the boundary Equation (10), Equation (11), and Equation (12).

$$z'' + \left(\lambda - \frac{C^2}{4\alpha^2}\right) z = 0, 0 < x < a, \tag{10}$$

$$z(0) = 0, \tag{11}$$

$$z(a) = 0. \tag{12}$$

The solution to the problem posed in the Equation (10) to Equation (12) is given by the Equation (13).

$$z_n(x) = \sin\left(\frac{n\pi x}{a}\right), \tag{13}$$

therefore, the solution of the Equation (6) to Equation (8) is given by the Equation (14).

$$v_n(x) = e^{\frac{Cx}{2\alpha}} z_n(x) = e^{\frac{Cx}{2\alpha}} \sin\left(\frac{n\pi x}{a}\right). \tag{14}$$

The solution to the function $w(t)$ comes from the Equation (15).

$$w_n(t) = e^{-\lambda_n \alpha t}, \tag{15}$$

where $\lambda_n = \frac{c^2}{4\alpha^2} + \frac{n^2\pi^2}{a^2}$. When replacing Equation (14) and Equation (15) in Equation (5) it is possible to calculate the function H_n, defined by the Equation (16).

$$H_n(x,t) = e^{\frac{Cx}{2\lambda}} e^{-\lambda_n \alpha t} \sin\left(\frac{n\pi x}{a}\right), \tag{16}$$

due to the principle of superposition the temperature function is the Equation (17).

$$H(x,t) = \sum_{n=1}^{\infty} b_n e^{\frac{Cx}{2\lambda}} e^{-\lambda_n \alpha t} \sin\left(\frac{n\pi x}{a}\right), \tag{17}$$

where the Fourier coefficient [12] b_n is defined by the Equation (18)

$$b_n = \frac{2}{a}\int_0^a f(x)\, e^{\frac{Cx}{2\lambda}} \sin\left(\frac{n\pi x}{a}\right) dx. \tag{18}$$

2.2. Numerical solution

The mathematical basis of the explicit numerical method [13] for solving the mathematical modeling of the transport Equation (1) is based on the representation of the Equation (19) by an approximation.

$$\frac{H_{i,j+1} - H_{i,j}}{k} = \alpha \frac{H_{i+1,j} - 2H_{i,j} + H_{i-1,j}}{(h)^2} - C\frac{H_{i+1,j} - H_{i,j}}{k}. \tag{19}$$

When dividing the spatial interval $(0, a)$ into equal parts, the magnitude h represents the width of each subdivision, likewise the k magnitude represents the width in the spatial interval $(0, t)$. The symbol $H_{i,j}$ in Equation (19) stand for the temperature $H(ih, jk)$. By means of an algebraic procedure in Equation (19) it is possible to express $H_{i,j+1}$ by the Equation (20).

$$H_{i,j+1} = \left(\frac{\alpha k}{h^2} - \frac{Ck}{h}\right)\left(H_{i+1,j}\right) + \left(1 - \frac{2\alpha k}{h^2} + \frac{Ck}{h}\right)\left(H_{i,j}\right) + \frac{\alpha k}{h^2} H_{i-1,j}. \tag{20}$$

3. Results and discussion

In order to validate the the solution of the transport Equation (1) by using the analytical method described in the Equation (17), we propose the mathematical system described in the Equation (21), Equation (22), Equation (23) and Equation (24).

$$\frac{\partial H}{\partial t} = 2\frac{\partial^2 H}{\partial x^2} - \frac{\partial H}{\partial x}, 0 < x < 1, \tag{21}$$

$$H(0,t) = 0, \tag{22}$$

$$H(1,\pi) = 0, \tag{23}$$

$$H(x,0) = -2e^{\frac{x}{4}}\sin(3\pi x), \tag{24}$$

By calculating the functions $v_n(x)$ and $w_n(t)$ described in the Equation (14) and Equation (15), using the mathematical framework of section 2.1, it is possible to compute the equivalent of the Equation (16) described in detail in the Equation (25).

$$H(x,t) = \sum_{n=1}^{\infty} b_n e^{\frac{x}{4} - \frac{16n^2\pi^2 t + t}{8}} \sin(n\pi x). \tag{25}$$

Matching Equation (24) and Equation (25) when $t = 0$, leds the Equation (26).

$$H(x,0) = \sum_{n=1}^{\infty} b_n e^{\frac{x}{4}} \sin(n\pi x) = -2e^{\frac{x}{4}}\sin(3\pi x), \tag{26}$$

from which $b_3 = -2$, $b_n = 0$ $(n \neq 3)$. Therefore, the function H has the form of the Equation (27).

$$H(x,t) = -2e^{\frac{x}{4} - \frac{16n^2\pi^2 t + t}{8}} \sin(3\pi x). \tag{27}$$

For the case study defined in this section in the differential Equation (21), the numerical scheme Equation (20), has the structural form of the Equation (28).

$$H_{i,j+1} = \left(\frac{2k}{h^2} - \frac{k}{h}\right)\left(H_{i+1,j}\right) + \left(1 - \frac{4k}{h^2} + \frac{k}{h}\right)\left(H_{i,j}\right) + \frac{2k}{h^2} H_{i-1,j}. \tag{28}$$

By using of computational simulation [13], it is possible to compare the analytical solution Equation (27) and the numerical scheme Equation (28), to establish the fit of both methods. Figure 1 presents two curves that calculate the temperature profile after 500 iterations.

The Figure 1 shows the behavior of the temperature profile in relation to the numerical and analytical methods; from the two curves it is possible to establish that the numerical method fits with to the analytical solution. Due to the choice of constant physical [14,15] parameters α and C throughout the computational simulation of the Equation (28), it is possible to conclude the good behavior of the curves in Figure 1. Table 1 displays the calculation of the relative error of the numerical method in relation to the analytical solution in order to demonstrate the fit of Figure 1 with a small error.

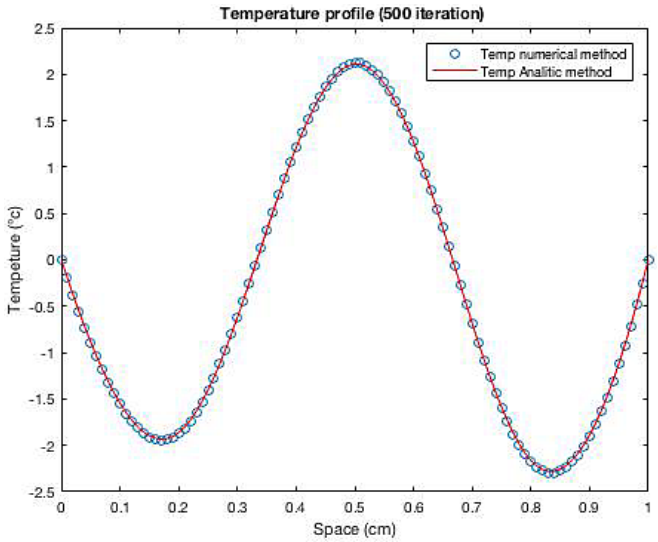


Figure 1. Contour of temperature with the numerical method and analitic method.

Table 1. Relative error of the numerical method.

Space (cm)	Relative error (500^{th}iteration)
0.0	0.0000
0.1	0.0104
0.2	0.0127
0.3	0.0044
0.4	0.0081
0.5	0.0143
0.6	0.0088
0.7	0.0045
0.8	0.0146
0.9	0.0129
1.0	0.0000

4. Conclusion

The research allowed modeling the transport phenomenon in the presence of convection with equations from mathematical physics. This approach allowed the application of differential equations techniques together with numerical methods to generate the solution of the mathematical modelling; by means of simulation it was possible to compare the analytical and numerical solutions, to conclude the fit of both methods.

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